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Jadie Adams, Gabriel Lopez, Casey Mann & Nhi Tran

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ARTICLES

Your Friendly Neighborhood Voderberg Tile

JADIE ADAMS
Westminster College
Salt Lake City, UT 84105
jadieraeadams@gmail.com

GABRIEL LOPEZ
California State University, San Bernardino
San Bernardino, CA 92407
gabriel.lopez-1@colorado.edu

CASEY MANN
University of Washington, Bothell
Bothell, WA 98011
cemann@uw.edu

NHI TRAN
University of Washington, Bothell
Bothell, WA 98011
vitamin933@gmail.com

In memory of Branko Grünbaum.

In 1934, K. Reinhardt asked if any tile exists such that two copies can completely enclose a third copy [4]. In 1936, Reinhardt's student Heinz Voderberg answered this question in the affirmative by inventing a special shape, now known as the *Voderberg tile*, shown in Figure 1. In fact, the two copies of the Voderberg tile can enclose* a third and a fourth copy (Figure 1)! More generally, a tile T has the *r-enclosure property* if two copies T_1 and T_2 of T can be arranged so that the complement of $T_1 \cup T_2$ has a bounded component, the closure of which is the union of r non-overlapping copies of T (as discussed, for example, by Grünbaum and Shepherd [3, 4]). In Figure 1, we see that the Voderberg tile has the 1- and 2-enclosure property.

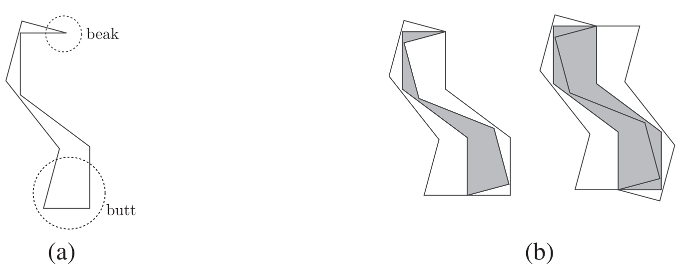

Figure 1 (a) The Voderberg tile. (b) The Voderberg tile has the 1-enclosure and the 2-enclosure property.

The enclosure property of the Voderberg tile is surprising and somewhat unintuitive, but this gangly tile's true beauty emerges in the tilings it generates, such as those shown in Figures 2(a) and 2(b). In this paper, we will explain how to construct a generalized

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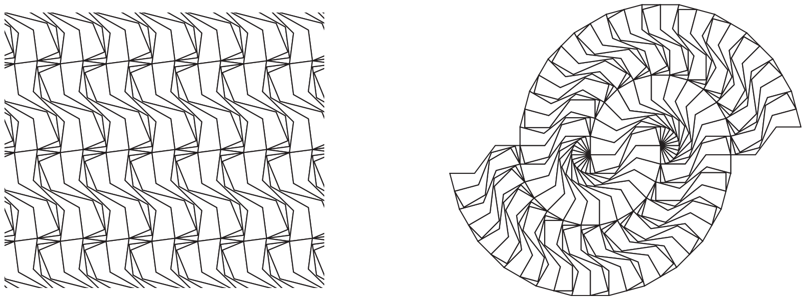
*Well, almost—see [5].

version of the tile in such a way that it can tessellate the plane, and we will go on to demonstrate some additional surprising properties that these tilings have.



(a) (b)

Figure 2 (a) A periodic tiling by the Voderberg tile. (b) A spiral tiling by the Voderberg tile.

Construction of a generalized Voderberg tile

The Voderberg tile was brought to a wide audience by Gardner [**1**], in 1977. Grünbaum and Shephard [**4**] describe a construction attributed to Goldberg [**2**] for a generalized family of tiles having members with the r-enclosure property for any $r \geq 1$. We will call these tiles *generalized Voderberg tiles* and denote them by V_n. For $n > 1$, these tiles are similar to the original Voderberg tile V_1, with the main differences being that they are thinner and that their "beaks" (see Figure 1) have n bends (or sections) instead of the single section of the original. Here we will reproduce and expound upon the construction in Grünbaum and Shephard [**4**] of the generalized Voderberg tile family.

Let n and k be positive integers with $k \gg 1$. Our goal is to describe the construction of V_n tiles with beak angle $\beta = \pi/k$. Having a beak angle that is commensurate with π allows for more kinds of spiral tilings to be formed from copies of the V_n tile, such as that in Figure 2. First we will give the construction of V_n that does not require β to be commensurate with π, then afterward we show how V_n with such specified β is obtained.

In the illustration of the construction in Figure 3 we use $n = 3$, but the construction described here is for general n. Start with 4 horizontal, parallel lines $a_1, a_2, a_3,$ and a_4 spaced 1 unit apart, then construct the polygonal line $ABCD$ with $A \in a_3$, $B \in a_1$, $C \in a_4$, and $D \in a_2$ with right angles at B and C. Let α be the acute angle formed by AB and a_1. Let DE be the circular arc between a_2 and a_4 centered at A, and similarly let AF be the circular arc between a_1 and a_3 centered at D. Let EE' be $1/n$ of the arc DE, and divide the arc AF into n equal subarcs

$$AA_1, A_1A_2, \ldots, A_{n-1}F.$$

Let S be the polygonal line

$$AA_1A_2 \ldots A_{n-1}FBCE$$

and let S' be the result of rotating S about A until E coincides with E' (i.e., through an angle of $-\theta/n$). The tile bounded by S, S', and EE' is then a generalized Voderberg tile V_n. Note that V_n is a $(2n+7)$-gon, and also observe that the beak angle β satisfies $\beta = \theta/n$ where $\theta = m(\angle DAE)$. In terms of the construction, the *beak* of the V_n tile is the thin part of the tile extending from F and F' to A, and the *butt* is the region near the segment EE'.

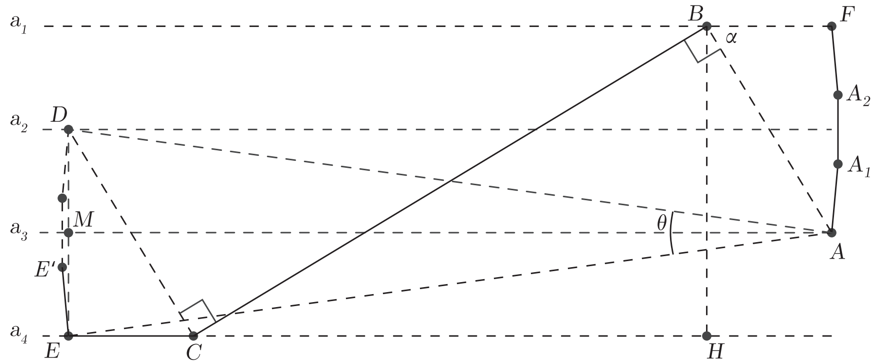


Figure 3 Construction of the generalized Voderberg tile.

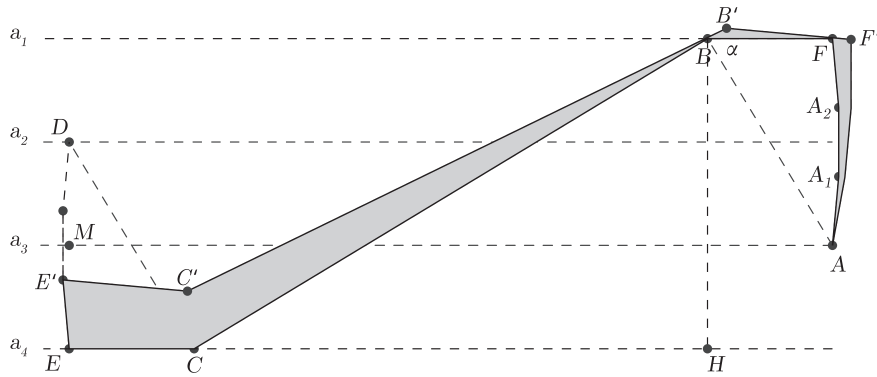


Figure 4 Completed construction of a V_3 tile.

The angle $\alpha = \angle ABF$ in Figure 3 was chosen arbitrarily in the construction described above, but with some care we can specify α so that the beak angle β is a factor of π. To that end, let $M \in a_3$ be the midpoint of DE and let H be the point on the a_4 such that BH is perpendicular to BF. Notice that

$$\triangle AFB, \quad \triangle BHC, \quad \text{and} \quad \triangle CED$$

are all similar right triangles, since

$$\angle ABF = \angle CBH = \angle DCE = \alpha,$$

with $DE = AF = 2$ and $BH = 3$. Now

$$EC = 2\cot\alpha, \quad CH = 3\tan\alpha, \quad \text{and} \quad BF = 2\cot\alpha.$$

It follows that

$$AM = EC + CH + BF = 4\cot\alpha + 3\tan\alpha.$$

Now, in $\triangle AMD$ we see that $\angle DAM = \theta/2$, and so

$$\tan\theta/2 = \frac{1}{4\cot\alpha + 3\tan\alpha} = \frac{\tan\alpha}{4 + 3\tan^2\alpha}. \tag{1}$$

Set $T = \tan(\theta/2)$ and $x = \tan\alpha$ so that equation (1) gives

$$3Tx^2 - x + 4T = 0. \tag{2}$$

We solve equation (2) for x to get

$$\tan\alpha = \frac{1 \pm \sqrt{1 - 48\tan^2(\theta/2)}}{6\tan(\theta/2)}. \tag{3}$$

Now require that $\beta = \theta/n = \pi/k$, so that $\theta/2 = n\pi/2k$. Then we must have

$$1 - 48\tan(\theta/2) = 1 - 48\tan\frac{n\pi}{2k} \geq 0.$$

Solving this inequality for k gives

$$k \geq \frac{n\pi}{2\tan^{-1}\left(\frac{1}{\sqrt{48}}\right)}. \tag{4}$$

Thus, based on n alone we can determine the range of allowable values of k.

For example, for a V_2 tile, we must have

$$k > \frac{\pi}{\tan^{-1}\left(\frac{1}{\sqrt{48}}\right)} \approx 21.9.$$

Let us choose $k = 25$. To construct a V_2 tile with beak angle $\beta = \pi/k = \pi/25$, we have

$$T = \tan\frac{n\pi}{2k} = \tan(\pi/25).$$

Thus, T in equation (2) is specified, and solving for x yields two roots x_1 and x_2, from which we determine two values of α:

$$\alpha = \tan^{-1} x_1 \approx 34.2611^\circ,$$
$$\alpha = \tan^{-1} x_2 \approx 62.9389^\circ.$$

Either of these values of α can be used to construct a V_2 with beak angle $\pi/25$, as in Figures 5 and 6. Notice that both shapes have the same length (from beak to butt). This is because in equation (1), the denominator of the first fraction is the length of AM, which depends only on θ. This was held constant in solving equation (2) for x.

In Figure 7, we see a V_2 tile exhibiting the 2-, 3-, and 4-enclosure property. Indeed, V_n tiles have the r- enclosure property for $r = 2n - 2$, $r = 2n - 1$, and $r = 2n$, which can be seen by noticing that that three enclosure values correspond to the manner in which the beaks meets the butts on the two enclosing tiles. Observe how the gray tiles meet in Figure 7. For general V_n, corresponding arrangements can be formed for the

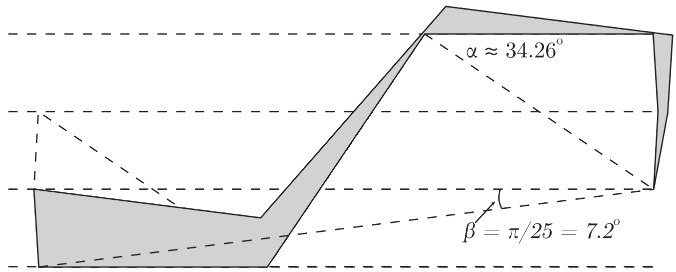


Figure 5 A V_2 tile with $k = 25$ and beak angle $\beta = \pi/25$ and $\alpha \approx 34.2611°$

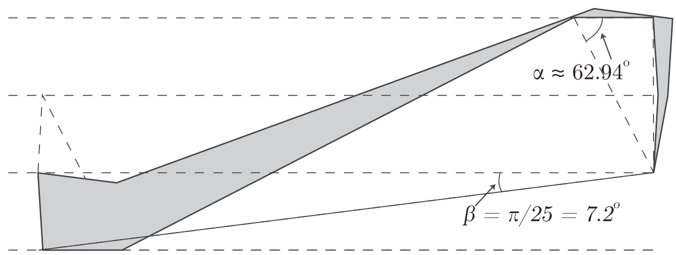


Figure 6 A V_2 tile with $k = 25$ and beak angle $\beta = \pi/25$ and $\alpha \approx 62.9389°$

enclosing (gray) tiles. Because of the number of beak segments and how the beak angle relates to the butt angle (at point E in the construction—that angle is exactly β more than a right angle), there will be room for exactly $2n - 2$, $2n - 1$, and $2n$ copies (respectively) in the middle, between the enclosing tiles.

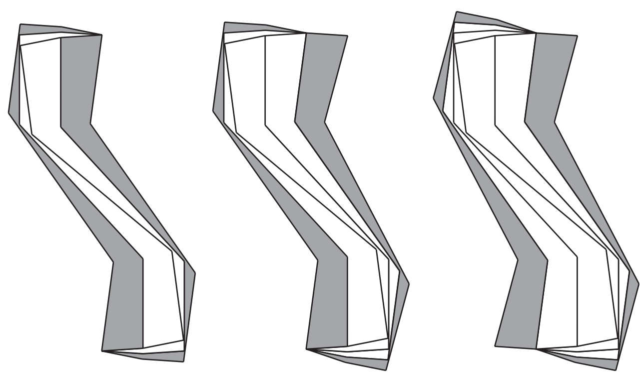


Figure 7 A V_2 tile exhibiting the 2-, 3-, and 4-enclosure property.

Problems from "G&S"

The contributions of Branko Grünbaum and G. C. Shephard to the development of a coherent and rigorous theory for tilings cannot be overstated, and much of their work is summarized in their magnum opus *Tilings and Patterns* [3]. In this work (and in others such as [4]) they were also generous in sharing open problems. In particular, these Johnny Appleseeds of tiling theory left two open problems for us which may be answered using the V_n tiles. Before giving these problems and their solutions, we need to go over some terminology.

A *tile* is a topological disk in the Euclidean plane, and a *tiling* of the plane is a countable collection of tiles whose interiors are pairwise disjoint and whose union is the plane. A tiling in which all of the tiles are congruent to one another is a *monohedral* tiling. Two tiles of a tiling are *neighbors* if their intersection is nonempty, and the *neighborhood* $\mathscr{N}(T)$ of a tile T in a tiling is the collection of neighbors of T (including T itself). For example, the neighborhood of a square in a standard edge-to-edge tiling by squares is that square and the eight squares surrounding it. It is possible that the union of the tiles in $\mathscr{N}(T)$ can fail to be simply connected, though it is difficult to imagine this occurring in a monohedral tiling. The *patch* $\mathscr{A}(T)$ generated by a tile T in a tiling is $\mathscr{N}(T)$ together with the minimal collection of tiles necessary to form a simply connected union.

With this terminology, we are ready to state the problems.

1. Decide whether for some (or for each) $r \geq 3$ there exists a tile T having the r-enclosure property and which admits a tiling of the plane (see Grünbaum and Shephard [**3**, p. 129] and [**4**]).
2. Show that if $\mathscr{T}$ is a monohedral tiling and T is a tile of $\mathscr{T}$, then $\mathscr{N}(T) = \mathscr{A}(T)$ (see Grünbaum and Shephard [**4**, p. 26]).

Solution to Problem 1. Problem 1 is readily solved by the V_n tiles. In particular, if the beak angle of a V_n tile is a factor of π, then such V_n tiles admit spiral tilings of the plane. As an added bonus, V_n can admit spiral tilings *while demonstrating the r-enclosure property*! The V_n tile also admits periodic tilings of the plane similar to that in Figure 2 while demonstrating the r-enclosure property.

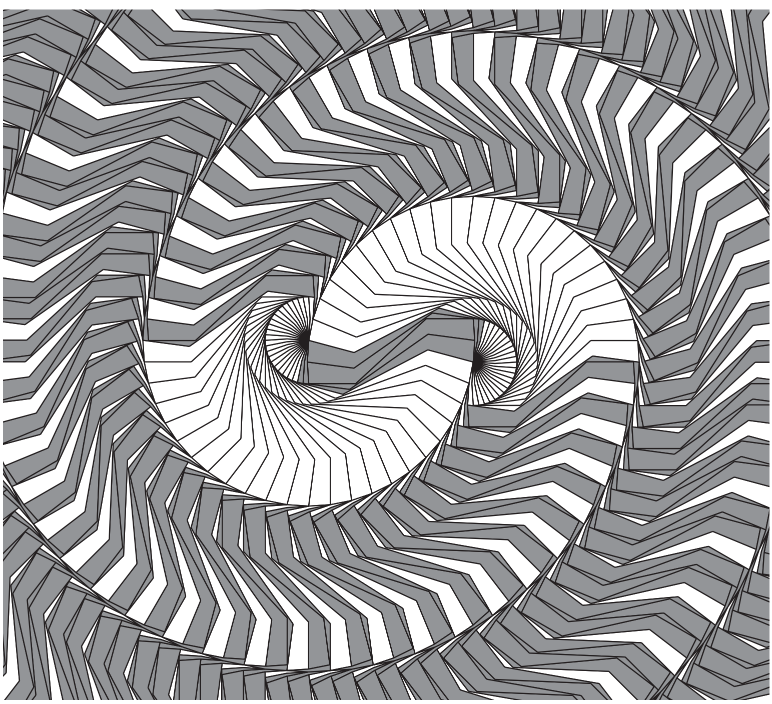


Figure 8 Spiral tiling admitted by the V_2 tile while displaying 4-enclosure property at the center.

Construction of the double spiral tiling by V_N

1. Begin at the center of the tiling by enclosing the desired amount of prototiles within two tiles.
2. To make the first layer of the spiral, fill in the two vertices where the enclosing tiles meet with beaks of tiles.
3. To make the second layer, move around the first layer adding a beak to butt pair of tiles followed by a single tile oriented so the beak is against the first layer.
4. To make the third layer, continue spiraling by adding two beak to butt pairs, then a single tile oriented so the beak is against the first layer.
5. The next layers are built in this same fashion. If you are adding layer k, first add $k-1$ beak to butt pairs and then the single tile. One can continue tiling outward in this fashion infinitely so that the entire plane is covered.

Solution to Problem 2. We provide a counterexample to show that the assertion of Problem 2 is false: In Figure 9, let T be the tile shaded in black. We see that $\mathscr{N}(T) \neq \mathscr{A}(T)$ because the tile shaded black is a member of $\mathscr{A}(T)$, but is not a member of $\mathscr{N}(T)$ (the tiles shaded gray are in both $\mathscr{N}(T)$ and $\mathscr{A}(T)$).

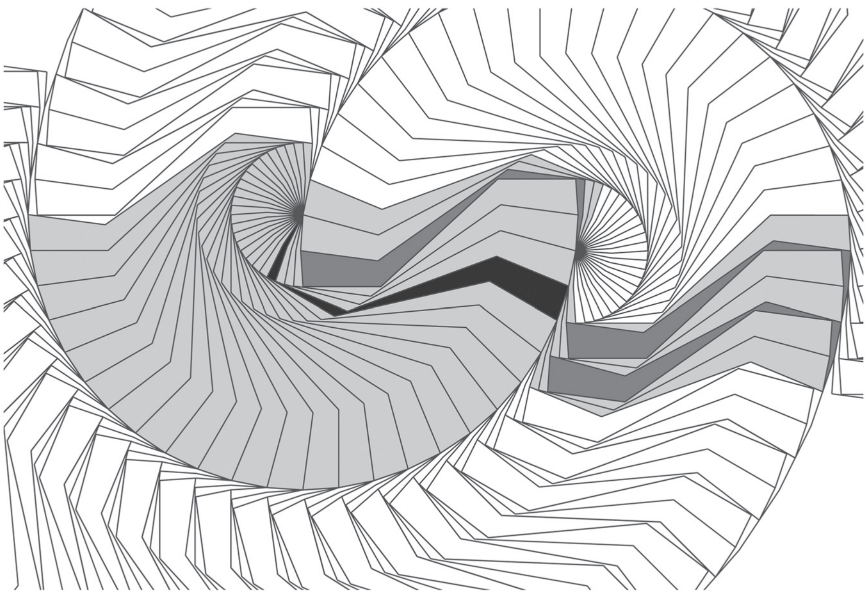


Figure 9 $\mathscr{A}(T) \neq \mathscr{N}(T)$. T is the black tile. The darker gray tiles are members of $\mathscr{A}(T)$ but not of $\mathscr{N}(T)$. The tiles shaded light gray are in both $\mathscr{N}(T)$ and $\mathscr{A}(T)$.

Discussion and open questions

We do not know what kind of witchcraft H. Voderberg wielded to invent his self-surrounding shape, but we do wonder if any tile with the r-enclosure property can be in any substantial way different from the Voderberg tile. Its construction allows for "self-spooning" (because of the rotated side), and we observe that this is critical to the behavior of the tiles.

We close with some open questions:

1. Is there a three-dimensional analog of the Voderberg tile? That is, is there a 3-D tile such that two copies can completely enclose some number of copies without gaps?

2. Is there an essentially different kind of 2-D tile that admits tilings of the plane while having the self-surrounding properties of the Voderberg tile?
3. Is there a 2-D tile such that two copies can *completely* surround (as an annulus) some number of copies while tiling the plane? In Mann [5], we see a modification of the Voderberg tile for which two copies can completely surround a number of copies, but that shape does not tile the plane.
4. In Grünbaum and Shephard [4, p. 129], we find a generalization of the r-enclosure property along with an open problem which we paraphrase here: A tile T has the (m, n)-enclosure property if two copies T_1 and T_2 of T can be arranged so that the complement of $T_1 \cup T_2$ has m bounded components, the closure of which is the union of n nonoverlapping copies of T. The open problem is: For each pair of positive integers m and n, find a tile with the (m, n)-enclosure property.

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Summary. We present a generalization of the Voderberg tile, which, in addition to admitting periodic and nonperiodic spiral tilings of the plane, has the property that just two copies can surround 1 or 2 copies of the tile. We construct a generalization of this tile that admits periodic and nonperiodic spiral tilings of the plane while enjoying the property that any number of copies of the tile can be surrounded by just 2 copies. In doing so, we solve two open problems posed in the classic book *Tilings and Patterns* by Grünbaum and Shephard.

JADIE ADAMS earned her Bachelor of Science degree in math from Westminster College before going on to work in automatic speech recognition. She is currently working toward a Ph.D. in computing at the University of Utah, where she her research focuses on machine learning and statistical shape modeling for medical image analysis.

GABRIEL LOPEZ earned his Bachelor of Science in mathematics at California State University, San Bernardino. As of this publication, he is a graduate student in mathematics at the University of Colorado, Boulder. He hopes to use his training to not only go into higher education, but to work in outreach and create research opportunities for students who are interested in the mathematical sciences, in particular to those from historically underrepresented groups in the community.

CASEY MANN is a mathematics professor at the University of Washington Bothell. His research interests include tilings and knot theory. He thinks it is impactful to engage undergraduate students in the process of mathematical discovery, as exemplified in this article and the accomplishments of his coauthors!

NHI TRAN earned her Bachelor of Science in Mathematics from the University of Washington Bothell in 2017.